Multivariable Optimization Multivariable Calculus X. Du

- **Maximum and minimum (local and global)**
- **Critical points**: points where ∇f is the zero vector or does not exist. Possible points for:
    - Extrema
    - Saddle points
- **Second Derivatives Test** for a function of two variables $f(x,y)$:
    - Let the discriminant $D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2$
    - If $D(a,b) > 0$ and either $f_{xx}(a,b) > 0$ or $f_{yy}(a,b) > 0$, then $f(a,b)$ is a local minimum.
    - If $D(a,b) > 0$ and either $f_{xx}(a,b) < 0$ or $f_{yy}(a,b) < 0$, then $f(a,b)$ is a local maximum.
    - If $D(a,b) < 0$, then $f(a,b)$ is a saddle point.
    - If $D(a,b) = 0$, then the second derivatives test is inconclusive.
- **Max-Min Theorem** for a function of two variables $f(x,y)$: Let *R* be a closed, bounded region in the xy-plane and let $f(x,y)$ be continuous on *R*. Then $f(x,y)$ has both an absolute maximum an absolute minimum on *R*, such that the absolute maximum and absolute minimum occur at a critical point inside *R* or on the boundary ∂R.
    - Check your extrema on your boundaries by substituting your boundary ∂R into your function or by using Lagrange multipliers.
- **Lagrange Multipliers**
    - Optimize a function within constraints.
    - A function of *n* variables must have $n-1$ constraints.
        - Note: That leaves 1 degree of freedom.
    - Example: optimize $f(x,y)$ within the constraint $g(x,y) = k$. The extreme value of $f(x,y)$ on this boundary will occur at a point (x_o, y_o) such that $\nabla f(x_o, y_o) = \lambda \nabla g(x_o, y_o)$.
    - Example: optimize $f(x,y,z)$ within the constraints $g(x,y,z) = k$ and $h(x,y,z) = c$. The extreme value of $f(x,y,z)$ given this constraint will occur at a point (x_o, y_o, z_o) such that $\nabla f(x_o, y_o, z_o) = \lambda \nabla g(x_o, y_o, z_o) + \mu \nabla h(x_o, y_o, z_o)$.
    - Finally, solve a systems of equations to find the potential points which will yield extrema.
    - Note: when optimizing the distance from the origin to a surface $z = f(x,y)$, it is much easier to optimize distance squared using Lagrange multipliers.